Avalanche exponents and corrections to scaling for a stochastic sandpile

R. Dickman* and J. M. M. Campelo
Departamento de Física, ICEx, Universidade Federal de Minas Gerais, Caixa Postal 702, 30161-970 Belo Horizonte, Minas Gerais, Brazil

(Received 6 January 2003; published 20 June 2003)

We study distributions of dissipative and nondissipative avalanches in Manna's stochastic sandpile, in one and two dimensions. Our results lead to the following conclusions: (1) avalanche distributions, in general, do not follow simple power laws, but rather have the form $P(s) \sim s^{-\tau_s}(\ln s)^\gamma f(s/s_c)$, with f a cutoff function; (2) the exponents for sizes of dissipative avalanches in two dimensions differ markedly from the corresponding values for the Bak-Tang-Wiesenfeld (BTW) model, implying that the BTW and Manna models belong to distinct universality classes; (3) dissipative avalanche distributions obey finite-size scaling, unlike in the BTW model.

DOI: 10.1103/PhysRevE.67.066111 PACS number(s): 05.65.+b, 05.90.+m, 05.40.−a

I. INTRODUCTION

Sandpile models are the prime examples of self-organized criticality (SOC) [1,2], a control mechanism that forces a system with an absorbing-state phase transition to its critical point [3,4], leading to scale invariance in the apparent absence of parameters [5]. Of central interest in sandpiles are avalanche distributions, which are expected to exhibit scale invariance, and the associated critical exponents. It is generally assumed that avalanche size and duration distributions follow simple power laws in the infinite-size limit, and that departures from such power laws reflect finite-size effects. (The finite system size leads to a rapid cutoff of the distribution beyond a characteristic avalanche size s_c; there are also corrections to power-law scaling at the small-s end of the distribution.) Such effects complicate the determination of critical exponents, since the estimates are sensitive to the choice of fitting interval.

Recently, Drossel showed that in the two-dimensional (2D) Bak-Tang-Wiesenfeld (BTW) sandpile, distributions of *dissipative* avalanches (in which one or more particles leave the system), follow clean power laws [6]. Nondissipative avalanche distributions must also follow power laws in the infinite-size limit [6], but are subject to much stronger corrections to scaling. The avalanche exponents for the two cases are very different, and the proportion of dissipative avalanches decreases $\sim L^{-1/2}$ with increasing system size L. Thus power-law fits to the total avalanche distribution represent a superposition of two distinct scaling behaviors (with L-dependent weights), and would appear to have no fundamental significance.

In light of these findings, it is of interest to study dissipative and nondissipative events separately in the stochastic sandpile as well. Our principal results are that avalanche distributions for Manna's sandpile are in general *not* pure power laws, but rather include a logarithmic correction, and that the dissipative avalanche exponents for the Manna model are quite different from those for the BTW model. The latter serves to resolve the issue of distinct universality classes for the two models, which has attracted considerable attention [7,8]. A further distinction between the two models concerns finite-size scaling, which is obeyed in the Manna model, but not (for dissipative avalanches) in the BTW sandpile [6]. In the following section we define the model. Simulation results are analyzed in Sec. III, and Sec. IV presents a brief summary of our results, and of open questions.

II. MODEL

The version of the Manna sandpile [9] studied here is defined on a hypercubic lattice with open boundaries: a chain of L sites in one dimension, a square lattice of $L \times L$ sites in 2D. The configuration is specified by the number of particles z_i, at each site i; sites with $z_i \geq 2$ are *active*, and have a toppling rate of unity. When site i topples, two particles move to randomly chosen nearest neighbors j and j' of i. (j and j' need not be distinct.) In one dimension we report results for $L=500$, 1000, 2000, 5000, 10 000, and 20 000 sites; in two dimensions the linear system sizes are $L=160$, 320, 640, 1280, and 2560. For the largest system sizes our results are based on samples of about 10^5 avalanches (10^6 in two dimensions), while for the smallest systems about 10^7 avalanches are generated. We study the model using both parallel and sequential updating; the results for the two dynamics show no systematic differences.

The sequential (continuous-time), Markovian dynamics consists of a series of toppling events at individual sites. The next site to topple is chosen at random from a list of active sites, which is updated following each event. The time increment associated with each toppling is $\Delta t = 1/N_A$, where N_A is the number of active sites just prior to the event. (Δt is the mean waiting time to the next event, if we were to choose sites blindly, instead of using a list. In this way, N_A sites topple per unit time, consistent with each active site having a unit rate of toppling.) When there are no active sites in the system, a particle is inserted at a randomly selected site, initiating a new avalanche. In the case of parallel dynamics all active sites release two particles simultaneously. The particle transfers define the configuration for the next time step.

*Email address: dickman@fisica.ufmg.br

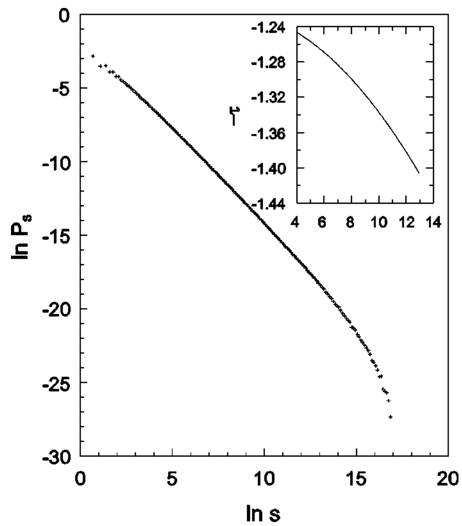

FIG. 1. Main graph: avalanche size distribution for the two-dimensional model, $L=1280$; the data points are simulation results and the superposed smooth curve a cubic fit to the data on the interval $4<\ln s<13$. Inset: estimate for the slope $-\tau$ of the cubic fit, versus $\ln s$.

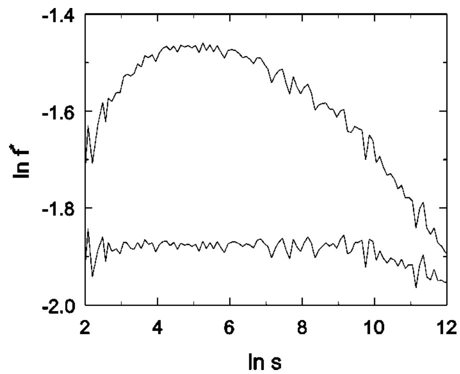

FIG. 2. Plot of $f^*=s^\tau P_s(s)/(\ln s)^\gamma$ versus $\ln s$ for the data shown in Fig. 1. Lower curve, best fit for $3<\ln s<10$ using $\tau=1.385$ and $\gamma=0.672$; upper curve, pure power-law fit using $\tau=1.25$.

III. RESULTS

We study the distributions $P_s(s)$ and $P_d(t)$ of avalanche sizes s and durations t; "size" means the number of topplings in an avalanche (the same site may contribute more than once to the size). The data are binned to equal intervals of $\ln s$ and $\ln t$. With increasing system size, an ever larger fraction of the avalanches are nondissipative; the fraction of dissipative avalanches decays $\sim L^{-1/2}$ in two dimensions. (For $L=2560$, only 2% of avalanches are dissipative.) In one dimension, however, the decay appears to be slower, roughly $\sim L^{-1/4}$.

The morphology of avalanche distributions in sandpiles generally includes a plateaulike region for small s or t, and a rapidly decaying portion for large events; between these limiting regimes there is a power-law-like interval. (The oscillations that sometimes appear at small s can be understood in terms of a parity effect that makes small avalanches more likely to have an *even* number of topplings.) The power-law interval is expected to grow with system size, so that the probability distribution in the second and third regimes follows:

$$P_s(s)=s^{-\tau_s}f_s(s/s_c), \tag{1}$$

where f_s is a scaling function that decays rapidly when its argument is $\geqslant 1$, and the cutoff size $s_c \sim L^{D_s}$. For $s \ll s_c$, the scaling function takes a constant value f_0. [A similar expression is anticipated for $P_d(t)$.] Analysis usually consists in selecting (in a plot of P_s versus s on log scales) a reasonably linear portion, and performing a linear regression to the data to determine τ_s.

Figure 1 illustrates the problematic nature of this procedure. We have plotted the size distribution of nondissipative avalanches in two dimensions (with the parallel update scheme), along with a polynomial fit to the data; the derivative of the latter, $-\tau$, is shown in the inset. Evidently we may have our choice of τ values ranging from 1.25 to 1.41!

Any hope of extracting simple, power-law scaling from a distribution of the kind shown in Fig. 1 (which is, in fact, typical of both size and duration distributions for nondissipative avalanches, regardless of system size or dimension), hinges on finding a suitable correction to scaling term. A natural choice, based on experience with critical phenomena, would be to include a factor of form $(1-as^{-\Delta})$ on the right-hand side (rhs) of Eq. (1). Attempts to fit such a form to the data consistently yield values of Δ very near zero, suggesting instead a logarithmic correction to scaling, so that Eq. (1) becomes

$$P_s(s)=s^{-\tau_s}(\ln s)^\gamma f_s(s/s_c). \tag{2}$$

This expression may be further generalized by writing the logarithmic term as $\ln(s/s_0)$; for reasons explained below, we have set $s_0=1$ in the present analysis.

We find that good fits to nondissipative avalanche distributions can only be obtained including the logarithmic correction. Our analysis consists in (1) making a preliminary estimate of the fitting interval $[x_0,x_1]$ (here $x\equiv\ln s$); (2) adjusting parameters τ and γ so as to minimize the variance of $f^*=s^\tau P_s(s)/(\ln s)^\gamma$ on the interval (ideally f^* would be constant and the variance zero); (3) checking for any systematic trend in f^*, and refining the fitting interval accordingly. In practice, we use the largest possible interval, excluding the small-s plateau regime and the large-s cutoff. For each kind of distribution, we use the same x_0 for each system size, while x_1 increases linearly with $\ln L$.

Figure 2 shows the result of this analysis, using $\tau=1.386$ and $\gamma=0.683$, for the data shown in Fig. 1. f^* fluctuates about a constant value over the optimum fitting interval, which in this case turns out to be $[2.8,10.3]$. (The derivative τ evaluated as in Fig. 1 varies between 1.18 and 1.31 on this interval.) For comparison we show the result of a pure power-law fit using the estimate $\tau_s=1.25$ [8]. The latter yields a strongly curved function $f^*(x)$, showing the inadequacy of a simple power law.

TABLE I. Best-fit parameters for the distribution of sizes of nondissipative avalanches in two dimensions. The final line gives estimated values for $L\to\infty$.

L	τ	γ
160	1.570	1.252
320	1.486	1.027
640	1.425	0.821
1280	1.389	0.696
2560	1.364	0.592
Ext.	1.302(10)	0.356(10)

The best-fit parameters for sizes of nondissipative avalanches in two dimensions are listed in Table I, with the final row indicating the result of an extrapolation to infinite size (the data fall close to a straight line when plotted versus $L^{-1/2}$). The parameters vary considerably with L, but in a systematic manner. Our estimate $\tau_{s,n}=1.30(1)$ (here the subscript denotes size, nondissipative; figures in parentheses denote uncertainties) is consistent with previous estimates of 1.28(2) [9], 1.25(2) [8], 1.27(1) [7], and 1.28(1) [10]. (Note that these studies include both dissipative and nondissipative avalanches in the analysis, which leads to a smaller exponent estimate, since τ is smaller for dissipative avalanches than for nondissipative ones.) Another important conclusion from the data in Table I is that the logarithmic correction *does not disappear* as $L\to\infty$. The asymptotic avalanche distribution, while scale invariant, does not follow a simple power law.

The duration distribution for nondissipative avalanches in 2D, and both size and duration distributions in 1D, follow the pattern described above. In each instance the best-fit parameters τ and γ decrease systematically with L, leading to the exponent estimates listed in Table II. Also listed are the exponents D governing the mean size and duration, defined via $\overline{s_n}\sim L^{D_{s,n}}$ (and similarly for the mean avalanche duration $\overline{t_n}\sim L^{D_{d,n}}$).

For *dissipative* avalanches, some intriguing differences appear. In 2D, the size distributions can be fit to high accuracy using $\gamma=0.5$, and the duration distributions using $\gamma=1$. Thus the logarithmic correction shows no significant size dependence. This is reminiscent of the observation of clean power laws for dissipative avalanches (but not for nondissipative ones) in the BTW model [6]. In 1D, *no* logarithmic correction is required to fit the dissipative avalanche distributions. (In all other cases, γ approaches a nonzero limiting value as $L\to\infty$). Finally, the best-fit values for τ show much less size dependence than in the nondissipative case. For avalanche sizes in 2D, for example, we find $\tau=1.004$, 0.988, 0.985, 0.978, and 0.975, for $L=160,\ldots,2560$, respectively. In one dimension the exponent estimates for dissipative avalanches show no systematic size dependence. Exponent values for dissipative avalanches, including $D_{s,d}$ and $D_{d,d}$, are listed in Table III.

TABLE II. Best estimates for exponents associated with distributions of sizes (s) and durations (d) of nondissipative avalanche distributions in one and two dimensions.

Case	τ	γ	D
s, 1D	1.11(2)	0.9(2)	1.92(1)
d, 1D	1.18(2)	1.45(10)	1.23(1)
s, 2D	1.30(1)	0.36(1)	1.94(2)
d, 2D	1.55(4)	0.85(6)	0.72(1)

TABLE III. Best estimates for exponent τ associated with distributions of sizes (s) and durations (d) of dissipative avalanche distributions in one and two dimensions, with γ fixed at the indicated value. Note the absence of a logarithmic correction in 1D.

Case	τ	γ	D
s, 1D	0.637(2)	0	2.20(1)
d, 1D	0.465(5)	0	1.47(1)
s, 2D	0.98(2)	1/2	2.74(6)
d, 2D	0.965(5)	1	1.42(1)

We have also performed a fitting analysis allowing the value of s_0 [mentioned in the discussion following Eq. (2)] to vary. While inclusion of an additional parameter leads to marginally improved fits, the best-fit values of s_0 do not differ greatly from unity, and follow no systematic trend with system size. (In the nondissipative case we have also tried fixing $\gamma=2$ or 3, and allowing s_0 to vary. This leads to essentially the same quality of fit as with $s_0=1$ and γ variable.) In summary, we find no advantage, given the present data, in including s_0 as a further adjustable parameter.

An important consequence of our results is that the Manna model belongs to a different universality class than the BTW sandpile. This follows by comparing the two-dimensional Manna value, $\tau_{s,d}=0.98(2)$, with the known value of 7/9 for BTW [6]. Note also that the exponent governing the mean size of dissipative avalanches is $D_{s,d}=2.74(6)$, while this exponent is equal to 2 for the BTW model [6]. Our results strengthen the conclusion reached by Biham *et al.* [8], on the basis of SOC sandpiles, and by Vespignani *et al.* [11], who studied "fixed energy" sandpiles having closed boundaries and strictly conserved particle

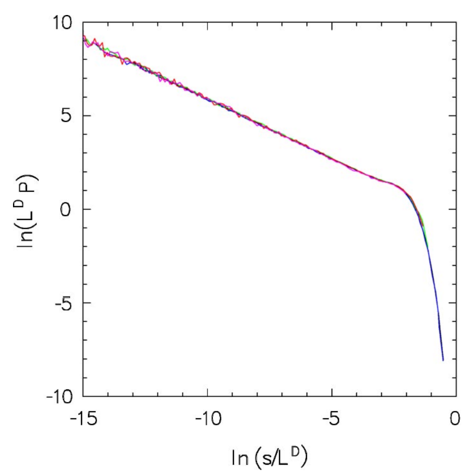

FIG. 3. Finite-size scaling plot of dissipative avalanche size distributions in one dimension, $L=500,\ldots,10^4$.

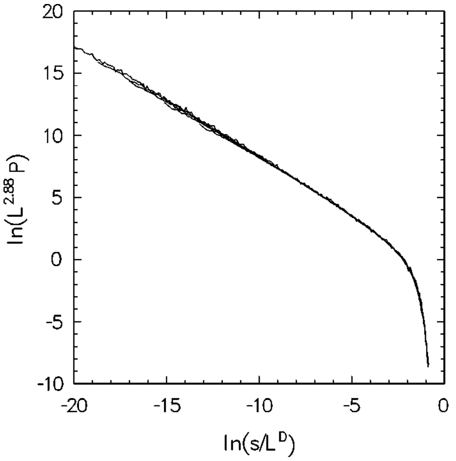

FIG. 4. Finite-size scaling plot of dissipative avalanche size distributions in two dimensions, $L=160, \ldots, 2560$.

number. The difference in universality classes is difficult to distinguish on the basis of total avalanche distributions [7], due to the similarity of the exponents for nondissipative avalanches in the two models.

In the 2D case, exponents $\tau_{s,d}$ and $\tau_{d,d}$ are so similar, and so near unity, that one might conjecture that they are both equal to 1. It is therefore useful to determine the relation between sizes and durations of dissipative avalanches, which is expected to follow a power law, $t \sim s^{x_d}$. A study using $L=2560$ yields $x_d=0.57(1)$. Recalling the scaling relation $x=(1-\tau_s)/(1-\tau_d)$ [9], we see that $\tau_{d,d}<\tau_{s,d}$ in this case. (The values quoted in Table III yield $x_d=0.6(6)$; the uncertainty is too great to permit a meaningful comparison.)

In the one-dimensional case, $\tau_{d,d}$ and $\tau_{s,d}$ are sufficiently different from unity that a quantitative comparison is possible. Simulations ($L=10^4$) yield $x_d=0.681(5)$, while $(1-\tau_s)/(1-\tau_d)=0.682(6)$. For nondissipative avalanches the comparisons are, in 1D, $x_n=0.692(5)$ (simulation) and 0.47(27) (scaling); in 2D, $x_n=0.593(5)$ (simulation), 0.55(6) (scaling).

In Figs. 3 and 4 we show size distributions for dissipative avalanches in one and two dimensions, respectively. They are more linear over a larger interval, compared with the nondissipative case. In the 1D system, the terminus of the power-law regime is signaled not by a decaying probability, but rather by a slight excess of large events. This excess of large dissipative events appears to be necessary for the 1D system to maintain a stationary mean particle density. Since $\tau<1$ for dissipative avalanches (as is also the case for the BTW model [6]), distributions P_s and P_d will not be normalizable if the scaling function f in Eq. (2) attains an L-independent limiting value as $L\to\infty$. In fact, we find that $f_0 \sim L^{-0.8}$ in one dimension, and $\sim L^{-0.2}$ in 2D.

A further important difference between the BTW and Manna sandpiles concerns finite-size scaling (FSS), which was shown to be violated for dissipative avalanches in the BTW model [6]. By contrast, we find that FSS holds for dissipative avalanches in the Manna model, as shown in Figs. 3 and 4. (That the total avalanche distribution obeys FSS has been demonstrated many times [9,10].) In each case, we achieve a clean data collapse by scaling the size s to its mean, which is proportional to $L^{D_{s,d}}$. In one dimension, the probability must be multiplied by $L^{D_{s,d}}$ to ensure collapse, so that the avalanche distribution has the scaling form $P_s(s)=L^{-D}\mathcal{P}(L^{-D}s)$. In two dimensions, however, the probability must be multiplied by $L^{2.88}$ to achieve a collapse.

Finally, we have examined the form of the large-s cutoff (or large-t cutoff, in the case of the duration), by studying $f=s^{\tau}P_s(s)/(\ln s)^{\gamma}$. This function is well approximated by $f=A\exp(-[(s/s_c)+b(s/s_c)^2])$.

IV. SUMMARY

We find that for Manna's stochastic sandpile, simple power-law avalanche distributions are the exception rather than the rule, and are observed only for nondissipative avalanches in the one-dimensional system. In all other cases a logarithmic correction is present. (Since our conclusions are based entirely on simulation data, we cannot rule out other correction to scaling forms, but it is evident that the correction decays very slowly.) Our data indicate that the correction persists in the infinite-size limit.

Analyzing the exponent estimates for a series of system sizes, we obtain estimates for the exponents τ that are consistent with scaling and (in the 2D nondissipative case) with previous simulations. Our results for $\tau_{s,d}$ and $D_{s,d}$ clearly place the Manna and BTW models in distinct universality classes. Our study highlights the importance of analyzing dissipative and nondissipative avalanches separately.

Our results raise several questions regarding avalanche distributions. First, what is the physical origin and theoretical basis for the logarithmic correction? Second, do such corrections appear in other models exhibiting SOC? Finally, which are the essential features of the Manna and BTW models leading to their rather different scaling properties? We hope to investigate these issues in future work.

ACKNOWLEDGMENTS

We thank Deepak Dhar, Alessandro Vespignani, and Kim Christensen for helpful comments. This work was supported by CNPq and CAPES, Brazil.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).

[2] D. Dhar, Physica A **263**, 4 (1999), and references therein.

[3] R. Dickman, M.A. Muñoz, A. Vespignani, and S. Zapperi, Braz. J. Phys. **30**, 27 (2000); e-print cond-mat/9910454.

[4] M.A. Muñoz, R. Dickman, R. Pastor-Satorras, A. Vespignani, and S. Zapperi, in *Modeling Complex Systems*, edited by J. Marro and P.L. Garrido, AIP Conf. Proc. No. 574 (AIP, Melville, NY, 2001); e-print cond-mat/0011442.

[5] G. Grinstein, in *Scale Invariance, Interfaces and Nonequilib-*

rium Dynamics Vol. 344 of *NATO Advanced Studies Institute, Series B: Physics*, edited by A. McKane *et al.* (Plenum, New York, 1995) Vol. 344.
[6] B. Drossel, Phys. Rev. E **61**, R2168 (2000).
[7] A. Chessa, H.E. Stanley, A. Vespignani, and S. Zapperi, Phys. Rev. E **59**, R12 (1999).
[8] O. Biham, E. Milshtein, and O. Malcai, Phys. Rev. E **63**, 061309 (2001).
[9] S.S. Manna, J. Phys. A **24**, L363 (1991); J. Stat. Phys. **59**, 509 (1990).
[10] R. Pastor-Satorras and A. Vespignani, Eur. Phys. J. A **19**, 583 (2001).
[11] A. Vespignani, R. Dickman, M.A. Muñoz, and S. Zapperi, Phys. Rev. E **62**, 4564 (2000).